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# Subgroups of classical groups normalized by relative elementary groups

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ABSTRACT

Let $(R, \Lambda)$ be a commutative form ring, and let $(J, \Gamma)$ be a form ideal of $(R, \Lambda)$. We obtain a complete description of all subgroups of the unitary groups $\mathrm{U}_{2n}(R, \Lambda)$ which are normalized by relative elementary subgroup $\mathrm{EU}_{2n}(J, \Gamma)$ for all $n \geq 4$.



## 1. Introduction

Let $R$ be a commutative ring with identity, $\mathrm{GL}_n(R)$ be the group of invertible $n$ by $n$ matrices over $R$, $\mathrm{E}_n(R)$ be the subgroup of $\mathrm{GL}_n(R)$ generated by all elementary matrices $e_{ij}(a) = 1_n + a\mathrm{E}_{ij}$, where $a \in R$, $\mathrm{E}_{ij}$ is a matrix with 1 in position $(i, j)$ and zeros elsewhere. For any ideal $I$ of $R$, let $\mathrm{C}_n(R, I)$ denote the full pre-image of the center of $\mathrm{GL}_n(R/I)$. Let $\mathrm{E}_n(I)$ be the subgroup of $\mathrm{GL}_n(R)$ generated by all $e_{ij}(a)$, where $a \in I$, and let $\mathrm{E}_n(R, I)$ denote the normal closure of $\mathrm{E}_n(I)$ in $\mathrm{E}_n(R)$.

A subgroup $H \subset G$ is called subnormal if there is a chain

$$H_d = H \lhd H_{d-1} \lhd \cdots \lhd H_0 = G \tag{1.1}$$

of subgroups of the group $G$, where $H_i \lhd H_{i-1}$ means that $H_i$ is a normal subgroup of $H_{i-1}$. If this is the case, we write $H \lhd^d G$.

The classification of the subnormal subgroups of $\mathrm{GL}_n(R)$ is related to the description of the subgroups of $\mathrm{GL}_n(R)$ normalized by $\mathrm{E}_n(R, J)$ (*i.e.*, generalized sandwich classification). Namely, if $G$ is a subgroup of $\mathrm{GL}_n(R)$ containing $\mathrm{E}_n(R)$ in (1.1), where $R$ is commutative, $n \geq 3$, then $H \lhd^d G$ implies that

$$\mathrm{E}_n(R, I^m) \subset H \subset \mathrm{C}_n(R, I) \tag{1.2}$$

for some ideal $I$ of $R$, where $m = f(d, n)$ is a function of $d$ and $n$ (see [12,13]) and $I^m$ denotes the ideal of $R$ consisting of sums of products of $m$ elements of $I$. From the 1970s to the early 1990s, Wilson [16], Bak [2], Li and Liu [10], Vavilov [15] and Vaserstein [12,13] presented and improved the bound of $m$ (see (1.2)) several times. One can see [8] for further reference. Now, we have the following theorem (see [13]).

**Theorem.** *Let $R$ be a commutative ring, $n \geq 3$ and $H$ a subgroup of $\mathrm{GL}_n(R)$ normalized by $\mathrm{E}_n(R, J)$ for an ideal $J$ of $R$. Thus, there exists an ideal $I$ of $R$ such that*

$$\mathrm{E}_n(R, I) \subset H \subset \mathrm{C}_n(R, (I : J^4)),$$

*where* $I : J^4 = \{r \in R \mid rJ^4 \subset I\}$.

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From the theorem above we obtain that $m = f(d, n) = (4^d - 1)/3$ for $n \geq 3$ (see (1.2)).

In [2], Bak conjectured that the generalized sandwich classification theorem holds as well for certain 'unitary' groups over rings with stable rank conditions. Habdank [5] settled Bak's conjecture positively with quadratic stable conditions and 2 invertible on the commutative ring. Zhang [17] proved the conjecture for stable unitary group $\mathrm{U}(R, \Lambda) = \lim_{n\to\infty} \mathrm{U}_{2n}(R, \Lambda)$ over commutative rings. Recently, Zhang [18] studied the non-stable case of unitary groups over a commutative ring with 2 invertible. In this paper, we answer Bak's conjecture positively for unitary groups $\mathrm{U}_{2n}(R, \Lambda)$ with $n \geq 4$ without any assumption regarding the commutative ring with identity.

## 2. Notation and main result

Let $R$ be an associative ring with identity 1, and assume that an anti-automorphism of order 2, *i.e.*, an involution $* : x \to x^*$ is defined on $R$ such that $(x + y)^* = x^* + y^*$, $(xy)^* = y^*x^*$, and $(x^*)^* = x$ for all $x, y$ in $R$. Clearly, $*$ also determines an anti-automorphism of the ring $\mathrm{M}_n(R)$ of all $n$ by $n$ matrices $(x_{ij})$ by $(x_{ij})^* = (x^*_{ji})$.

Fix an element $\lambda \in \mathrm{Cent}(R)$ such that $\lambda\lambda^* = 1$. Set $\Lambda_{\min} = \{x - x^*\lambda \mid x \in R\}$ and $\Lambda_{\max} = \{x \in R \mid x = -x^*\lambda\}$. A form parameter $\Lambda$ is an additive subgroup of $R$ such that

(1) $\Lambda_{\min} \subseteq \Lambda \subseteq \Lambda_{\max}$,
(2) $r^*\Lambda r \subseteq \Lambda$ for all $r \in R$.

The pair $(R, \Lambda)$ is called a form ring. For an involution invariant ideal $I$ of $R$, *i.e.*, $I = I^*$, define $\Gamma_{\max} = I \cap \Lambda$ and $\Gamma_{\min} = \{x - x^*\lambda \mid x \in I\} + \{x^*\xi x \mid \xi \in \Lambda, x \in I\}$.

A relative form parameter $\Gamma$ in $(R, \Lambda)$ of level $I$ is an additive subgroup of $I$ such that

(1) $\Gamma_{\min} \subseteq \Gamma \subseteq \Gamma_{\max}$,
(2) $r^*\Gamma r \subseteq \Gamma$ for all $r \in R$.

The pair $(I, \Gamma)$ is called a form ideal of the form ring $(R, \Lambda)$. We denote the set of $\Lambda$-anti-Hermitian matrices by

$$\mathrm{AH}_n(R, \Lambda) = \{(a_{ij}) \in \mathrm{M}_n(R) \mid a_{ij} = -a^*_{ji}\lambda \text{ for } i \neq j \text{ and } a_{ii} \in \Lambda,\ i = 1, \ldots, n\}.$$

Following Bak [1] (see also [14]), we define the unitary (or quadratic or generalized unitary) groups

$$\mathrm{U}_{2n}(R, \Lambda) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}_{2n}(R) \,\middle|\, \alpha^*\delta + \gamma^*\lambda\beta = 1_n, \alpha^*\gamma, \beta^*\delta \in \mathrm{AH}_n(R, \Lambda) \right\}.$$

When $\Lambda = \Lambda_{\max}$,

$$\mathrm{U}_{2n}(R, \Lambda_{\max}) = \{\theta \in \mathrm{GL}_{2n}(R) \mid \theta^*\varphi_n\theta = \varphi_n\},$$

where $\varphi_n = \begin{pmatrix} 0 & 1_n \\ \lambda 1_n & 0 \end{pmatrix}$.

In general, $\mathrm{U}_{2n}(R, \Lambda) \subseteq \mathrm{U}_{2n}(R, \Lambda_{\max})$; hence, $\theta^{-1} = \varphi_n^{-1}\theta^*\varphi_n$ for any $\theta \in \mathrm{U}_{2n}(R, \Lambda)$, and if $\theta = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ then

$$\theta^{-1} = \begin{pmatrix} \delta^* & \beta^*\lambda^* \\ \gamma^*\lambda & \alpha^* \end{pmatrix}. \tag{2.1}$$

A column $v = (v_1, \ldots, v_n)^T$ is called unimodular if there exist $b_1, \ldots, b_n \in R$ such that $b_1v_1 + \cdots + b_nv_n = 1$. For example, any column of an invertible matrix is unimodular.

Let $R^\times$ denote the group of invertible elements in $R$.

Fix an $n$, and for any $1 \leq k \leq 2n$ set $\sigma k = k + n$ if $k \leq n$ and $\sigma k = k - n$ if $k > n$. Furthermore, $\epsilon(i)$ denotes the sign of an integer $i$ defined as $\epsilon(i) = 1$ if $i \leq n$ and $\epsilon(i) = -1$ if $i > n$. We define two types of elementary matrices in unitary group as follows:

$$\rho_{i,\sigma i}(a) = 1_{2n} + a\mathrm{E}_{i,\sigma i} \tag{2.2}$$

with $a \in \lambda^{-(\epsilon(i)+1)/2}\Lambda$, *i.e.*, $a \in \lambda^*\Lambda$ when $i \leq n$ and $a \in \Lambda$ when $i > n$, which are called long root elements;

$$\rho_{ij}(a) = 1_{2n} + a\mathrm{E}_{ij} - \lambda^{(\epsilon(j)-\epsilon(i))/2}a^*\mathrm{E}_{\sigma j,\sigma i} \tag{2.3}$$

with $a \in R, j \neq i,\ \sigma i$, which are called short root elements.

We denote the subgroup of $\mathrm{U}_{2n}(R, \Lambda)$ generated by all elementary unitary matrices as $\mathrm{EU}_{2n}(R, \Lambda)$. For a form ideal $(I, \Gamma)$ of $(R, \Lambda)$, an elementary matrix $\rho_{ij}(a)$ is called elementary of level $(I, \Gamma)$ if $a \in I$ when $j \neq i,\ \sigma i$, and $a \in \lambda^{-(\epsilon(i)+1)/2}\Gamma$ when $j = \sigma i$. The subgroup of $\mathrm{U}_{2n}(R, \Lambda)$ generated by elementary matrices of level $(I, \Gamma)$ is denoted by $\mathrm{FU}_{2n}(I, \Gamma)$, and the normal subgroup of $\mathrm{EU}_{2n}(R, \Lambda)$ generated by $\mathrm{FU}_{2n}(I, \Gamma)$ is denoted by $\mathrm{EU}_{2n}(I, \Gamma)$.

Let $(I, \Gamma)$ be a form ideal of $(R, \Lambda)$. The principal congruence subgroup $\mathrm{U}_{2n}(I, \Gamma)$ of level $(I, \Gamma)$ in $\mathrm{U}_{2n}(R, \Lambda)$ is defined as

$$\mathrm{U}_{2n}(I, \Gamma) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{U}_{2n}(R, \Lambda) \;\middle|\; \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \equiv 1_{2n} (\mathrm{mod}\, I), \text{ and } \alpha^*\gamma, \beta^*\delta \in \mathrm{AH}_n(R, \Gamma) \right\},$$

whereas the full congruence subgroup $\mathrm{CU}_{2n}(I, \Gamma)$ is defined as

$$\mathrm{CU}_{2n}(I, \Gamma) = \{g \in \mathrm{U}_{2n}(R, \Lambda) \mid [g, \mathrm{U}_{2n}(R, \Lambda)] \subseteq \mathrm{U}_{2n}(I, \Gamma)\}.$$

It can be shown (see [4]) that $\mathrm{CU}_{2n}(I, \Gamma_{\max})$ is the full pre-image of the center of $\mathrm{U}_{2n}(R/I, \ \Lambda/\Gamma_{\max})$, *i.e.*,

$$\mathrm{CU}_{2n}(I, \Gamma_{\max}) = \varphi_I^{-1}(\mathrm{Center}(\mathrm{U}_{2n}(R/I, \ \Lambda/\Gamma_{\max}))),$$

where $\varphi_I$ denotes the canonical projection: $R \to R/I$.

From now on, we assume the ring $R$ is commutative.

**Definition 2.1** ([5]). Let $(I, \Gamma^I)$ and $(J, \Gamma^J)$ be two form ideals in $(R, \Lambda)$. The product of $(I, \Gamma^I)$ and $(J, \Gamma^J)$ is defined as $(IJ, \Gamma^I\Gamma^J)$, where

$$\Gamma^I\Gamma^J = \Gamma^{IJ}_{\min} + \{a^*ba \mid \ a \in J, b \in \Gamma^I\} + \{b^*ab \mid \ b \in I, a \in \Gamma^J\}.$$

**Definition 2.2.** Let $(I, \Gamma^I)$ and $(J, \Gamma^J)$ be two form ideals in $(R, \Lambda)$, and $k, \ l$ two non-negative integers. The parameter $\Omega(J, I, \Gamma^I, k, l)$ of the form ideal $(I : J^k, \Omega(J, I, \Gamma^I, k, l))$ is defined as

$$\Omega(J, I, \Gamma^I, k, l) = \Gamma^{I:J^k}_{\min} + \{\xi \in \Gamma^{I:J^k}_{\max} \mid \ a^*\xi a \in \Gamma^I, \text{ for all } a \in J^l\}.$$

Now, we state the main results of the current study.

**Theorem 1.** *Let $(R, \Lambda)$ be a commutative form ring, and let $(J, \Gamma^J)$ be a form ideal. Let $H$ be a subgroup of $\mathrm{U}_{2n}(R, \Lambda)$, $n \geq 4$, normalized by $\mathrm{EU}_{2n}(J, \Gamma^J)$. Thus, there exists a form ideal $(I, \Gamma^I)$ such that*

$$\mathrm{EU}_{2n}(I, \ \Gamma^I) \subseteq H \subseteq \mathrm{CU}_{2n}(I : J^{12}, \ \Omega(J, I, \Gamma^I, 12, 14)).$$

## 3. Elementary facts

In this section, we state and prove several preparatory lemmas concerning some properties of elementary subgroups of $\mathrm{U}_{2n}(R, \Lambda)$.

**Lemma 3.1** ([6,14]). *The following identities hold for elementary unitary matrices $(1 \leq i \neq j \leq 2n)$.*

(i) $\rho_{ij}(a + b) = \rho_{ij}(a)\rho_{ij}(b)$,
(ii) $[\rho_{ij}(a), \rho_{jk}(b)] = \rho_{ik}(ab)$ *when $i, j, k, \sigma i, \sigma j, \sigma k$ are all distinct,*
(iii) $[\rho_{ij}(a), \rho_{j,\sigma i}(b)] = \rho_{i,\sigma i}(ab - \lambda^{-\epsilon(i)}a^*b^*)$ *when $j \neq i, \ \sigma i$,*
(iv) $[\rho_{ij}(a), \rho_{j,\sigma j}(b)] = \rho_{i,\sigma j}(ab)\rho_{i,\sigma i}(\lambda^{(\epsilon(j)-\epsilon(i))/2}a^*ba)$ *when $j \neq i, \ \sigma i$ and $b \in \lambda^{-(\epsilon(i)+1)/2}\Lambda$.*

**Proposition 3.2** ([4,6,14]). *Let $n \geq 3$. Thus, for any form ideal $(I, \Gamma^I)$ of $(R, \Lambda)$*

(i) $\mathrm{EU}_{2n}(I, \Gamma^I)$ *is a normal subgroup of* $\mathrm{U}_{2n}(R, \Lambda)$,
(ii) $\mathrm{EU}_{2n}(I, \Gamma^I) = [\mathrm{EU}_{2n}(R, \Lambda), \mathrm{U}_{2n}(I, \Gamma^I)] = [\mathrm{EU}_{2n}(R, \Lambda), \mathrm{CU}_{2n}(I, \Gamma^I)]$.

**Lemma 3.3.** *Let $g \in \mathrm{U}_{2n}(R, \Lambda)$. Then, $g \in \mathrm{U}_{2n}(0 : J, \Gamma^{(0:J)}_{\max})$ if and only if $g\rho_{ij}(a) = \rho_{ij}(a)g$ for all $\rho_{ij}(a) \in \mathrm{EU}_{2n}(J, \Gamma^J_{\max})$. In turn, this is equivalent to the similar condition $g\rho_{ij}(a) = \rho_{ij}(a)g$ for all short root elements $\rho_{ij}(a)$ with $a \in J$ and $j \neq \sigma i$.*

**Proof.** The first assertion was proved in [17]. In fact, for any $g = (g_{ij}) \in \mathrm{U}_{2n}(R, \Lambda)$, $g$ commutes with all $\rho_{ij}(a) \in \mathrm{EU}_{2n}(J, \Gamma^J_{\max})$ if and only if

(i) $ag_{ij} = a^*g_{ij} = 0$ when $j \neq i$,
(ii) $ag_{ii} = ag_{jj}, a^*g_{ii} = a^*g_{jj}$ when $j \neq \sigma i$,
(iii) $ag_{ii} = ag_{\sigma i,\sigma i}, a \in \lambda^{-(\epsilon(i)+1)/2}\Gamma^J_{\max}$.

Taking $a \in \lambda^{-(\epsilon(i)+1)/2}\Gamma^J_{\max}$ in (ii), we have $ag_{ii} = ag_{jj} = ag_{\sigma i,\sigma i}$ by (ii). This implies that if $g$ commutes with all short root elements $\rho_{ij}(a)$ with $a \in J$, then $g$ commutes with all long root elements $\rho_{i,\sigma i}(a)$ with $a \in \lambda^{-(\epsilon(i)+1)/2}\Gamma^J_{\max}$. This proves the second assertion. $\square$

**Lemma 3.4.** *Let $H$ be a subgroup of $\mathrm{U}_{2n}(R, \Lambda)$, $n \geq 3$, normalized by $\mathrm{EU}_{2n}(J, \Gamma)$, and let $g \in H$. Let $t \in \mathrm{U}_{2n}(R, \Lambda)$ and $h_i \in \mathrm{EU}_{2n}(J, \Gamma)$, $i = 1, 2, \ldots, k$. Then,*

$$[\cdots[[tgt^{-1}, h_1], h_2], \ldots, h_k] \in {}^tH = tHt^{-1}.$$

**Proof.** Recall that by Proposition 3.2 $\mathrm{EU}_{2n}(J, \Gamma)$ is a normal subgroup of $\mathrm{U}_{2n}(R, \Lambda)$. If $[tgt^{-1}, h_1] = tg't^{-1} \in {}^tH$, then $t(t^{-1}h_2t)g'(t^{-1}h_2^{-1}t)t^{-1} \in {}^tH$. Therefore, we only need to show that $[tgt^{-1}, h_1] \in {}^tH$. Because $[tgt^{-1}, h_1] = tgt^{-1}h_1tg^{-1}t^{-1}h_1^{-1} = t(gt^{-1}h_1tg^{-1}t^{-1}h_1^{-1}t)t^{-1}$, and the term in the parentheses belongs to $H$, we obtain the conclusion. □

For every $g = (g_{ij}) \in \mathrm{U}_{2n}(R, \Lambda)$, $o(g)$ denotes the ideal of $R$ generated by all $g_{ij}$ and $g_{ii} - g_{jj}$. This ideal is usually called the level of $g$. In particular, $o(g) = o(g^{-1})$ and $o(g)$ is $*$-invariant, *i.e.*, $o(g)^* = o(g)$, by (2.1). The form parameter $\Gamma^{o(g)}$ in the level $o(g)$ is defined as

$$\Gamma^{o(g)} = \Gamma_{\min}^{o(g)} + \left\{ \sum_{i=1}^{n} g_{ij}^* g_{i+n,j} \mid j = 1, \ldots, 2n \right\} + \left\{ \sum_{j=1}^{n} \lambda g_{ij}^* g_{i,j+n} \mid i = 1, \ldots, 2n \right\}.$$

It is obvious that $\Gamma^{o(g)} = \Gamma^{o(g^{-1})}$ (see (2.1) ).

Let $v = (v_1, v_2, \ldots, v_{2n})^T \in R^{2n}$. The length $|v|_q$ of $v$ is defined as

$$|v|_q = v_1^* v_{n+1} + \cdots + v_n^* v_{2n} + \Lambda$$

where $\Lambda$ is the form parameter of $(R, \Lambda)$.

Denote the length of the $k$th column and the $k$th row of a matrix $g \in \mathrm{U}_{2n}(R, \Lambda)$ as $|v_{g(k)}|_q$ and $|u_{g(k)}|_q$, respectively. By the definition of $\mathrm{U}_{2n}(R, \Lambda)$, one has $|v_{g(k)}|_q \in \Lambda$. On the other hand, if $g \in \mathrm{U}_{2n}(I, \Gamma)$, then necessarily $o(g) \subseteq I$ and $|v_{g(k)}|_q \in \Gamma$.

**Lemma 3.5.** *Let $g = (g_{ij}) \in \mathrm{U}_{2n}(R, \Lambda)$ and $\rho = \rho_{ij}(a)$ with $j \neq \sigma i$. Denote by $\eta$ the commutator $[g, \rho]$. Thus, for $k \neq j$, $\sigma i$, one has*

$$|v_{\eta(k)}|_q = a^* g_{\sigma k, \sigma j} |v_{g(i)}|_q g_{\sigma k, \sigma j}^* a + a^* g_{\sigma k, i} |v_{g(\sigma j)}|_q g_{\sigma k, i}^* a + w,$$

*and for $k = j$ or $\sigma i$ one has*

$$|v_{\eta(k)}|_q = a^* |v_{\eta(l)}|_q a + a^* g_{\sigma k, \sigma j} |v_{g(i)}|_q g_{\sigma k, \sigma j}^* a + a^* g_{\sigma k, i} |v_{g(\sigma j)}|_q g_{\sigma k, i}^* a + w',$$

*where $l = i$ if $k = j$ and $l = \sigma j$ if $k = \sigma i$. In the above formulas, $w, w' \in \{g_{pq} a g_{rh} - \lambda g_{rh}^* a^* g_{pq}^*\}$, where $|v_{\eta(l)}|_q$ in the latter is defined by the former.*

**Proof.** Write $\eta$ as $(1_{2n} + a v_i \widetilde{v}_j - \lambda^{(\epsilon(j) - \epsilon(i))/2} a^* v_{\sigma j} \widetilde{v}_{\sigma i}) \rho_{ij}(-a)$, where $v_i$ is the $i$th column of $g$ and $\widetilde{v}_j = v_{\sigma j}^* \varphi_n$ is the $j$th row of $g^{-1}$, respectively. Then, the assertion follows from direct computation and definition of unitary groups. □

**Lemma 3.6.** *Let $H$ be a subgroup of $\mathrm{U}_{2n}(R, \Lambda)$, $n \geq 4$, normalized by $\mathrm{EU}_{2n}(J, \Gamma^J)$ and let $k$ be a non-negative integer. If $g \in H$ but $g \notin \mathrm{CU}_{2n}(I : J^k, \Gamma_{\max}^{(I:J^k)})$, then there exists at least one non-diagonal entry $g_{ij}$ of ${}^tg = tgt^{-1}$, where $t \in \mathrm{EU}_{2n}(R, \Lambda)$, such that $g_{ij}J^k \notin I$. Moreover, there is a short root element $\rho_{jl}(a)$ with $a \in J$ such that*

$$h = [{}^tg, \rho_{jl}(a)] \notin \mathrm{U}_{2n}(I : J^{k-1}, \Gamma_{\max}^{(I:J^{k-1})}).$$

**Proof.** By the definition of $\mathrm{CU}_{2n}(I : J^k, \Gamma_{\max}^{(I:J^k)})$ and $o(g)$, there exists at least one non-diagonal entry $g_{ij}$ of $g$ or $g_{ii} - g_{jj}$ in $o(g)$ that does not lie in $(I : J^k)$. If $g_{ij} \notin (I : J^k)$, we are done. Suppose that all non-diagonal entries of $g$ lie in $(I : J^k)$, but $g_{ii} - g_{jj} \notin (I : J^k)$ for some $i \neq j$. We claim that there is a $j \neq \sigma i$ such that $g_{ii} - g_{jj} \notin (I : J^k)$. Otherwise, if only $g_{ii} - g_{\sigma i, \sigma i} \notin (I : J^k)$ and all $g_{\sigma i, \sigma i} - g_{ll} \in (I : J^k)$, where $l \neq i$, $\sigma i$, then $(g_{ii} - g_{\sigma i, \sigma i}) - (g_{ll} - g_{\sigma i, \sigma i}) = g_{ii} - g_{ll} \notin (I : J^k)$, which is a contradiction. Conjugating $g$ by $t = \rho_{ij}(1)$, we obtain the matrix ${}^tg$, whose entry in the position $(i, j)$ equals $g_{ij} + (g_{ii} - g_{jj})$ and does not belong to $(I : J^k)$.

Now, suppose that there is a non-diagonal entry $g_{ij}$ in ${}^tg$ satisfying $ag_{ij} \notin (I : J^{k-1})$ for some $a \in J$. Then, taking $\rho = \rho_{jl}(a)$ where $l \neq i, \sigma i, \sigma j$ and comparing the entries of ${}^tg\rho$ and $\rho{}^tg$ in the position $(i, l)$, we have that $g_{il} + ag_{ij} \not\equiv g_{il} (\mathrm{mod}\ (I : J^{k-1}))$. Hence, $[{}^tg, \rho] \not\equiv 1_{2n} (\mathrm{mod}\ (I : J^{k-1}))$. Thus, $[{}^tg, \rho] \notin \mathrm{U}_{2n}(I : J^{k-1}, \Gamma_{\max}^{(I:J^{k-1})})$. □

**Remark 3.7.** (1) In Lemma 3.6, we showed that $[{}^tg, \rho_{jl}(a)] \notin \mathrm{U}_{2n}(I : J^{k-1}, \Gamma_{\max}^{(I:J^{k-1})})$. In fact $[{}^tg, \rho_{jl}(a)]$ does not belong to $\mathrm{CU}_{2n}(I : J^{k-1}, \Gamma_{\max}^{(I:J^{k-1})})$ either, because if $[{}^tg, \rho_{jl}(a)] \in \mathrm{CU}_{2n}(I : J^{k-1}, \Gamma_{\max}^{(I:J^{k-1})})$ then ${}^tg\rho_{jl}(a) = \rho_{jl}(a){}^tg\ (\mathrm{mod}\ (I : J^{k-1}))$. The proof is quite similar to that of Lemma 24 in [13]; thus, we omit it.

(2) It is easy to see that there is at least one non-diagonal entry $h_{ir}$ in the $i$th row of $h = [{}^tg, \rho_{jl}(a)]$ that does not lie in $(I : J^{k-1})$. Similarly, there is at least one non-diagonal entry $h_{m,\sigma i}$ in the $\sigma i$th column of $h$ not lying in $(I : J^{k-1})$. Otherwise, the diagonal entry $h_{ii}$ or $h_{\sigma i, \sigma i}$ of $h$ could be written as $1 + \beta$, where $\beta \notin (I : J^{k-1})$. This is because if the $i$th row $u$ of $h$ is congruent to $(0, \ldots, 1, 0, \ldots, 0)$ modulo $(I : J^{k-1})$, in comparing the entries of ${}^tg\rho_{ij}(a)$ and $h\rho_{ij}(a){}^tg$ in the position $(i, l)$ modulo $(I : J^{k-1})$, we may obtain a contradiction.

## 4. Localization

Denote the subring of $R$ generated by all $rr^*$ with $r \in R$ as $R_0$. In the following sections, all multiplicative systems considered will be in $R_0$. Meanwhile, we will mostly use localization with respect to the following two types of multiplicative systems.

(i) Principal localization: take $s \in R_0$, and set multiplicative system coincides with $\langle s\rangle = \{1, s, s^2, \dots, \}$. The localization of the form ring $(R, \Lambda)$ with respect to multiplicative system $\langle s\rangle$ is denoted by $(R_s, \Lambda_s)$.

(ii) Maximal localization: take $M \in \mathrm{Max}(R_0)$, the set of maximal ideals in $R_0$, and let $S = R_0\backslash M$. Localization of the form ring $(R, \Lambda)$ with respect to multiplicative system $S$ is denoted by $(R_M, \Lambda_M)$, *i.e.*, $R_M = S^{-1}R$, $\Lambda_M = S^{-1}\Lambda$.

For the two cases, we write $F_s : R \to R_s$ and $F_M : R \to R_M$, respectively, to denote the corresponding localization homomorphism.

Straightforward computations show that the pair $(R_s, \Lambda_s)$ and $(R_M, \Lambda_M)$ are form rings.

By [14, Lemma 1.4], the ring $R_M$ is semi-local with at most two maximal ideals for every $M \in \mathrm{Max}(R_0)$.

Denote the subset of $\mathrm{FU}_{2n}(R, \Lambda)$ consisting of all products of $m$ or fewer short root elements $\rho_{ij}(a)$ as $\mathrm{E}^m(R)$, where $a \in R$, one of the indices $i, j$ is fixed, *i.e.*, the products are $\prod_{1\le i\ne j,\sigma j\le 2n}^{m} \rho_{ij}(a)$, where $j$ is fixed.

The following lemma will be used in the proof of Proposition 6.3 which in turn is the key step in the proof of our main theorem.

**Lemma 4.1.** *Suppose $R$ is a semi-local ring with at most two maximal ideals and let $g \in \mathrm{U}_{2n}(R, \Lambda)$. Thus, there is a matrix $t \in \mathrm{E}^2(R)$ such that the first diagonal entry of ${}^t g$ is invertible. Furthermore, ${}^t g = uh$, where $u = \prod_{i=2}^{2n} \rho_{i1}(*)$, $h$ is a unitary matrix of the parabolic type, i.e., the first column of $h$ is of the form $(h_{11}, 0, \dots, 0, 0, \dots, 0)^T$.*

**Proof.** Let $M_1$, $M_2$ be the maximal ideals of $R$. Then, $J(R) = M_1 \cap M_2$ and $R/J(R) \cong F_1 \oplus F_2$, where $F_1$ and $F_2$ are fields. An element $\alpha$ in $R$ is invertible if and only if $\alpha \equiv (a, b) \pmod{J(R)}$ with $a, b \ne 0$. Now, suppose that the first diagonal entry $g_{11}$ of $g$ is not invertible. The proof can be subdivided into two cases: (i) $g_{11} \notin J(R)$, (ii) $g_{11} \in J(R)$. Below, we consider the first case; the second one is similar.

In case (i), we may assume that $g_{11} \equiv (a, 0) \pmod{J(R)}$ with $a \ne 0$. Because the rows and columns of an invertible matrix are unimodular, there is an entry $g_{1i}$ $(i \ne 1)$ in the first row of $g$ such that $g_{11} + g_{1i}x \in R^\times$ for some $x \in R$. If $i \ne n + 1$, take $t = \rho_{i1}(-x)$. On the other hand, if $i = n + 1$ and all other entries in the first row lie in the same maximal ideal $M_2$ as $g_{11}$, take $t = \rho_{21}(-y)\rho_{n+2,1}(-\lambda x^*)$, where $y \equiv (0, b) \pmod{J(R)}$ with $b \ne 0$.

The second conclusion of the lemma can be obtained easily by the first one. $\square$

**Lemma 4.2.** *For each $M \in \mathrm{Max}(R_0)$ fix an element $s \in R_0\backslash M$. Let $g \in \mathrm{U}_{2n}(R, \Lambda)$. Then $g \in \mathrm{CU}_{2n}(I, \Gamma^I_{\max})$ if and only if $F_s(g) \in \mathrm{CU}_{2n}(I_s, \Gamma^{I_s}_{\max})$ for all $M \in \mathrm{Max}(R_0)$.*

The proof is similar to Lemma 5 in [11] and Lemma 9 in [9]; thus, we omit it.

In general, the group homomorphisms $F_s : \mathrm{U}_{2n}(R, \Lambda) \to \mathrm{U}_{2n}(R_s, \Lambda_s)$ and $F_M : \mathrm{U}_{2n}(R, \Lambda) \to \mathrm{U}_{2n}(R_M, \Lambda_M)$ induced by the localization homomorphisms $(R, \Lambda) \to (R_s, \Lambda_s)$ and $(R, \Lambda) \to (R_M, \Lambda_M)$ are not injective. However, for principal localization of Noetherian rings, the restrictions of these homomorphisms to some sufficiently small congruence subgroups are injective.

**Lemma 4.3** (*See [7, Lemma 5.1] and [3, Lemma 4.10]*)**.** *Let $R$ be a Noetherian ring, $s \in R$. Then there exists a non-negative integer $p$ such that the homomorphism $F_s : \mathrm{U}_{2n}(s^pR, s^p\Lambda) \to \mathrm{U}_{2n}(R_s, \Lambda_s)$ is injective.*

For $\zeta \in R_s$, there exists a non-negative integer $N_\zeta$ such that $s^{N_\zeta}\zeta \in F_s(R)$. It follows that for a finite subset $L$ one can find a non-negative integer $N$ such that $s^N\zeta \in F_s(R)$ for all $\zeta \in L$. Thus, one can always choose an integer $p$ such that both the restriction of localization homomorphism $F_s : s^pR \to R_s$ is injective and $s^p\zeta \in F_s(R)$ for all elements $\zeta$ in a finite subset of $R_s$. In the following sections, we apply this, for instance, to the entries of a given unitary matrix.

The pair $(R_M, \Lambda_M)$ is the direct limit of pairs $(R_s, \Lambda_s)$, where $s \in S = R_0\backslash M$. For any functor $F$ commuting with direct limits, one has $F(R_M, \Lambda_M) = \varinjlim F(R_s, \Lambda_s)$. In particular, $\mathrm{U}_{2n}(R_M, \Lambda_M) = \varinjlim \mathrm{U}_{2n}(R_s, \Lambda_s)$. This simple argument reduces most questions regarding Noetherian rings. Similarly, instead of localizations with respect to arbitrary multiplicative systems, we can consider only principal localizations.

## 5. Extraction of short root elements

Throughout this section, $H$ is a subgroup of $\mathrm{U}_{2n}(R, \Lambda)$, $n \ge 4$, normalized by $\mathrm{EU}_{2n}(J, \Gamma^J)$, and $\langle s\rangle$ is a fixed multiplicative system such that $F_s(H) \not\subseteq \mathrm{CU}_{2n}(I_s : J_s^k, \Gamma_{\max}^{(I_s:J_s^k)})$. In Section 4 we have chosen an integer $p$ such that both the restriction of localization homomorphism $F_s : s^pR \to R_s$ is injective and $s^p\zeta \in F_s(R)$ for all $\zeta$ in a finite subset of $R_s$, which is considered. Thus, the induced homomorphism $F_s : \mathrm{EU}_{2n}(s^pJ, \Gamma^{s^pJ}) \to \mathrm{EU}_{2n}(J_s, \Gamma_s)$ is injective. Meanwhile, if $a$ is an element in $R_s$ such that $a \notin (I_s : J_s)$, then $s^pad \notin F_s(I)$ (does not lie in $I_s$ either) for some $d \in J_s$.

**Lemma 5.1.** *If $H$ contains a matrix $g$ such that the first column of $g$ is of the form $(1,0,\ldots,0,0,\ldots,0)^T$ and there is an entry $g_{ij}$ with $i\neq 1, j\neq n+1$ of $g$ satisfying that $g_{ij}\notin (I:J^k)$ where $j\neq i$ or $g_{ij}-1\notin (I:J^k)$ where $j=i$, then $H$ contains two short root elements $\rho_{1l}(a)$, $\rho_{1h}(a)$, where $h\neq l, \sigma l$ and $a\notin (I:J^{k-3})$.*

**Proof.** By the assumption and the definition of a unitary matrix , $g$ and $g^{-1}$ have the following matrix form

$$\begin{pmatrix} 1 & * & \cdots & * & * & * & \cdots & * \\ 0 & & & & * & & & \\ \vdots & & A_1 & & \vdots & & B_1 & \\ 0 & & & & * & & & \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & & & & * & & & \\ \vdots & & C_1 & & \vdots & & D_1 & \\ 0 & & & & * & & & \end{pmatrix} \tag{5.1}$$

Note that $g_{ij}$ is in fact an entry in $\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1\end{pmatrix}$. Choose $d_1\in J$ such that $g_{ij}d_1\notin (I:J^{k-1})$ when $j\neq i$, or $(g_{ij}-1)d_1\notin (I:J^{k-1})$ when $j=i$. Thus, the matrix $g_1=[g^{-1},\rho_{1i}(d_1)]\notin \mathrm{U}_{2n}(I:J^{k-1},\Gamma_{\max}^{(I:J^{k-1})})$ and $g_1$ can be written as $\prod_{i=2}^{2n}\rho_{1i}(*)$, where $\rho_{1j}(*)=\rho_{1j}(d_1g_{ij})$. Choose $d_2,d_3\in J$ such that $d_1g_{ij}d_2d_3\notin (I:J^{k-3})$. Thus, the two matrices $\rho_{1r_m}(d_1g_{ij}d_2d_3)=[[g_1,\rho_{jp}(d_2)],\rho_{p,r_m}(d_3)]$ (the lower index $m$ of $r$ takes 1 and 2), where $r_1\neq 1,p,\sigma p,j$ and $n+1$, $r_2\neq 1,p,\sigma p,j,r_1,\sigma r_1$ and $n+1$ (note that $2n\geq 8$); moreover, $p$ takes values from $\{2,\ldots,2n\}$ except $1,j,\sigma j$ and $n+1$, lie in $H$ but not in $\mathrm{U}_{2n}(I:J^{k-3},\Gamma_{\max}^{(I:J^{k-3})})$. □

**Remark 5.2.** When $n=4$, investigating the range of values $r_2$ in the above proof, we only can obtain that $H$ contains two short root elements satisfying the above condition. Corollary 5.5 answers the question regarding why we show that $H$ contains two short root elements, not only one.

From now on, we write $\rho_{ij}(s^pa)$, which indicates $\rho_{ij}(F_s(s^pa))$.

**Lemma 5.3.** *If $F_s(H)$ contains a matrix $F_s(g)=\rho_{i1}(a)\rho_{j1}(b)\rho_{n+1,1}(\xi)$, where $a,b\in F_s(R)$, $\xi\in F_s(\Lambda)$, and $a$ satisfies $s^pad\notin (I:J^{k-1})$ ($s^pad$ can be considered an element of $R$) for some $d\in J$, then $H$ contains short root elements $\rho_{k1}(s^pad)$, where $k\neq 1,i,\sigma i,\sigma j,n+1$.*

**Proof.** Because $F_s(g\rho_{n+1,1}(-\xi)\rho_{j1}(-b)\rho_{i1}(-a))=1_{2n}$, $g=\rho_{i1}(a)\rho_{j1}(b)\rho_{n+1,1}(\xi)\beta$, where $\beta\in \mathrm{CU}_{2n}(\mathrm{Ann}(s^L),\Gamma_{\max}^{\mathrm{Ann}(s^L)})$ for some positive integer $L$. The element $\beta$ commutes with every element of the form $\rho_{lh}(s^Lc)$, $c\in R$. Because we can choose $p\geq L$, taking $\rho_{ki}(s^pd)$, where $d\in J$, and $k\neq 1,i,\sigma i,\sigma j,n+1$, we conclude that $H$ contains

$$\rho_{k1}(s^pad)=[\rho_{ki}(s^pd),g]=[\rho_{ki}(s^pd),\rho_{i1}(a)\rho_{j1}(b)\rho_{n+1,1}(\xi)]$$

where $s^pad\notin (I:J^{k-1})$. □

**Lemma 5.4.** *Let $t_1=\prod_{i=2}^{2n}\rho_{i1}(c_i)$, $t_3=\prod_{i=2}^{2n}\rho_{i1}(b_i)$, where $b_i,c_i\in R_s$, and let $t_2=\rho_{13}(1)\rho_{31}(-1)\rho_{13}(1)$ (case (i)) or $t_2=\rho_{12}(-1)\rho_{21}(1)$ (case (ii)). If ${}^{t_3t_2t_1}F_s(H)$ contains a short root element $\rho_{1r}(a)$, where $r\neq 3,n+3$ for case (i) and $r\neq 2,n+2$ for case (ii), then $H$ contains $2n-3$ short root elements $\rho_{i1}(s^{3p}ad_1d_2d_3)$, where $d_i(i=1,2,3)\in J$, and $i$ takes values from $\{2,\ldots,2n\}$ except $n+1$, 3 (for case (i)) or 2 (for case (ii)).*

**Proof.** By Proposition 3.2 and Lemma 3.4, ${}^{t_3t_2t_1}F_s(H)$ contains $\rho_{ir}(s^pad_1)=[\rho_{i1}(s^pd_1),\rho_{1r}(a)]$, where $d_1\in J$, and $i\neq 1,n+1,r,\sigma r,3,n+3$ for case (i); $i\neq 1,n+1,r,\sigma r,2,n+2$ for case (ii). Because $t_3=\prod_{i=2}^{2n}\rho_{i1}(b_i)$, direct computation shows that ${}^{t_2t_1}F_s(H)$ contains ${}^{t_3^{-1}}\rho_{ir}(s^pad_1)=\rho_{i1}(s^pad_1b_r)\rho_{\sigma r,1}(-s^pad_1b_{\sigma i})\rho_{n+1,1}(\xi)\rho_{ir}(s^pad_1)$, where $\xi\in F_s(\Lambda)$.

Thus, ${}^{t_1}F_s(H)$ contains

(i) $\eta_1=\rho_{i3}(-s^pad_1b_r)\rho_{\sigma r,3}(s^pad_1b_{\sigma i})\rho_{n+3,3}(\xi)\rho_{ir}(s^pad_1)$ for case (i),

(ii) $\eta_2=\rho_{i2}(-s^pad_1b_r)\rho_{\sigma r,2}(s^pad_1b_{\sigma i})\rho_{n+2,2}(\xi)\rho_{ir}(s^pad_1)$ for case (ii).

In the following, we will show the conclusion only for case (i) because the arguments are the same for case (ii).

Again, ${}^{t_1}F_s(H)$ contains

$$[\eta_1,\rho_{r1}(s^pd_2)]=\rho_{i1}(s^{2p}ad_1d_2)\rho_{n+3,1}(-\lambda s^{2p}a^*d_1^*d_2b_{\sigma i}^*),\quad \text{where } d_2\in J,$$

thus, $F_s(H)$ contains

$$\rho_{i1}(s^{2p}ad_1d_2)\rho_{n+3,1}(-\lambda s^{2p}a^*d_1^*d_2b_{\sigma i}^*)\rho_{n+1,1}(\zeta),\quad \text{where } \zeta\in F_s(s^{2p}\Lambda s^{2p}).$$

Applying Lemma 5.3, we obtain that $H$ contains $\rho_{k1}(s^{3p}ad_1d_2d_3)$, where $d_3\in J$, and $k\neq 1,i,\sigma i,3,n+1$. Note that the index $i$ of $\rho_{ir}(s^pad_1)$ may take at least two distinct values from $\{2,\ldots,2n\}$ except $n+1,r,\sigma r,3,n+3$. Thus, choosing two suitable distinct values for $i$, we can conclude that $H$ contains $2n-3$ short root elements $\rho_{i1}(s^{3p}ad_1d_2d_3)$, where $i$ takes values from $\{2,\ldots,2n\}$ except $n+1$ and 3 (for case (ii), $i\neq n+1,2$). $\square$

**Corollary 5.5.** *Under the assumptions of Lemma 5.4, if ${}^{t_3t_2t_1}F_s(H)$ contains a matrix $g$ whose first column is of the form $(1,0,\ldots,0,0,\ldots,0)^T$ and whose entry $g_{ij}$ with $i\neq 1, j\neq n+1$ in the position $(i,j)$ satisfies that $g_{ij}\notin (I_s:J_s^k)$, where $j\neq i$ or $g_{ij}-1\notin (I_s:J_s^k)$ where $j=i$, then $H$ contains $2n-3$ short root elements $\rho_{i1}(a)$, where $a\notin (I:J^{k-6})$, and $i$ takes values from $\{2,\ldots,2n\}$ except $n+1$, and 3 or 2.*

**Proof.** By Lemma 5.1, we conclude that ${}^{t_3t_2t_1}F_s(H)$ contains two short root elements $\rho_{1r_1}(a)$, $\rho_{1r_2}(a)$, where $a\notin (I_s:J_s^{k-3})$, and $r_2\neq r_1,\sigma r_1$. Thus, there is at least one of the indices $r_1,r_2$ that does not take a value from $\{3,n+3\}$ and another that does not take a value from $\{2,n+2\}$. That is, ${}^{t_3t_2t_1}F_s(H)$ contains a short root element $\rho_{1r_1}(a)$ with $r_1\neq 3,n+3$, and a short root element $\rho_{1r_2}(a)$ with $r_2\neq 2,n+2$. Thus, applying Lemma 5.4, we obtain the conclusion that $H$ contains $2n-3$ short root elements $\rho_{i1}(b)$, where $b\notin (I:J^{k-6})$, and $i$ takes values from $\{2,\ldots,2n\}$ except for $n+1$, 3 or 2. $\square$

## 6. The core of the proof

The main step in proving the 'generalized sandwich theorem' (see Theorem 1) is to show that $H$ contains some short root elements $\rho_{ij}(a)$ and that $\mathrm{EU}_{2n}(aJ^k+a^*J^k,\Gamma_{\min}^{(aJ^k+a^*J^k)})\subseteq H$. In this section, we first show that $\mathrm{EU}_{2n}(aJ^2+a^*J^2,\Gamma_{\min}^{(aJ^2+a^*J^2)})\subseteq H$ under the assumption that $H$ contains $2n-3$ short root elements $\rho_{i1}(a)$ and then show that ${}^{t_3t_2t_1}F_s(H)$, where $t_i$ $(i=1,2,3)$ are the matrices of the form same as that in Lemma 5.4, contains a matrix of parabolic type.

**Lemma 6.1.** *Let $H$ be a subgroup of $\mathrm{U}_{2n}(R,\Lambda)$, $n\geq 4$, normalized by $\mathrm{EU}_{2n}(J,\Gamma^J)$. If $H$ contains $2n-3$ short root elements $\rho_{i1}(a)$, then*

$$\mathrm{EU}_{2n}(aJ^2+a^*J^2,\Gamma_{\min}^{(aJ^2+a^*J^2)})\subseteq H.$$

**Proof.** It is known that $\mathrm{EU}_{2n}(aJ^2+a^*J^2,\Gamma_{\min}^{(aJ^2+a^*J^2)})$ is generated by all $\rho_{ij}(x)\rho_{ji}(b)\rho_{ij}(-x)$ where $j\neq\sigma i$, $x\in R$ and $b\in aJ^2+a^*J^2$, and $\rho_{i,\sigma i}(\zeta)\rho_{\sigma i,i}(\xi)\rho_{i,\sigma i}(-\zeta)$ where $\zeta\in\Lambda$, and $\xi\in\{d_2d_1a-\lambda^{-\epsilon(i)}a^*d_1^*d_2^*\}+\{d_2d_1a\eta a^*d_1^*d_2^*\mid\eta\in\Lambda\}$ where $d_1,d_2\in J$. We need to only show that all $\rho_{ij}(x)\rho_{ji}(ad_1d_2)\rho_{ij}(-x)$ where $j\neq\sigma i$, and all $\rho_{i,\sigma i}(\zeta)\rho_{\sigma i,i}(\xi)\rho_{i,\sigma i}(-\zeta)$ lie in $H$.

Without loss of generality, we assume that all short root elements $\rho_{i1}(a)$ with $i\neq 3$ are contained in $H$.

By Lemma 3.1 the following elementary matrices of level $(aJ+a^*J,\Gamma_{\min}^{(aJ+a^*J)})$ are contained in $H$.

$$\rho_{k1}(ad_1)=[\rho_{ki}(d_1),\rho_{i1}(a)]\quad\text{for all } 2\leq k\neq n+1\leq 2n, \tag{6.1}$$

$$\rho_{ik}(ad_1)=[\rho_{i1}(a),\rho_{1k}(d_1)]\quad\text{for all } 2\leq k\neq i,\sigma i\leq 2n \text{ and } i\neq 1,3,n+1, \tag{6.2}$$

$$\rho_{i,\sigma i}(ad_1-\lambda^{-\epsilon(i)}d_1^*a^*)=[\rho_{i1}(a),\rho_{1,\sigma i}(d_1)]\quad\text{where } i\neq 1,3,n+1, \tag{6.3}$$

$$\rho_{n+1,1}(ad_1-\lambda d_1^*a^*)=[\rho_{n+1,j}(d_1),\rho_{j1}(a)]\quad\text{where } j\neq 1,3,n+1. \tag{6.4}$$

Now, it is easy to show that $\mathrm{FU}_{2n}(aJ^2+a^*J^2,\Gamma_{\min}^{(aJ^2+a^*J^2)})\subseteq H$ by applying Lemma 3.1 and the above four formulas. For instance, $\rho_{1i}(ad_1d_2)=[\rho_{1j}(d_2),\rho_{ji}(ad_1)]$ for all $i$ except $i=n+1$, $\rho_{3k}(ad_1d_2)=[\rho_{3j}(d_2),\rho_{jk}(ad_1)]$ for all $k$ except $k=n+3$, $\rho_{1,n+1}(ad_1d_2-\lambda^*d_1^*d_2^*a^*)=[\rho_{1j}(d_2),\rho_{j,n+1}(ad_1)]$ and so on.

The rest of the proof involves showing that $\mathrm{EU}_{2n}(aJ^2+a^*J^2,\Gamma_{\min}^{(aJ^2+a^*J^2)})\subseteq H$. We do so only for short root elements because the proof for long root elements is similar.

Choosing $k\neq 1,3,n+1,i,j,\sigma i,\sigma j$ (note that $n\geq 4$), we have

$$\begin{aligned}
&\rho_{ij}(x)\rho_{ji}(ad_1d_2)\rho_{ij}(-x)\quad(x\in R,d_1,d_2\in J)\\
=&\rho_{ij}(x)[\rho_{jk}(d_2),\rho_{ki}(ad_1)]\rho_{ij}(-x)\\
=&[\rho_{ij}(x)\rho_{jk}(d_2)\rho_{ij}(-x),\rho_{ij}(x)\rho_{ki}(ad_1)\rho_{ij}(-x)].
\end{aligned}$$

Because $\rho_{ij}(x)\rho_{ki}(ad_1)\rho_{ij}(-x)=\rho_{kj}(-xad_1)\rho_{ki}(ad_1)\in H$ and $\rho_{ij}(x)\rho_{jk}(d_2)\rho_{ij}(-x)\in\mathrm{EU}_{2n}(J,\Gamma^J)$, we have that $\rho_{ij}(x)\rho_{ji}(ad_1d_2)\rho_{ij}(-x)\in H$. $\square$

**Remark 6.2.** If $H$ contains only one short root element, then we can obtain that $\mathrm{EU}_{2n}(aJ^4+a^*J^4,\Gamma_{\min}^{(aJ^4+a^*J^4)})\subseteq H$ (see [17,18]). However, in general, the lower the exponent $k$ of the ideal $J$ is, the better the final result will be.

**Proposition 6.3.** *Suppose that $H$ is a subgroup of $\mathrm{U}_{2n}(R,\Lambda)$, $n\geq 4$, normalized by $\mathrm{EU}_{2n}(J,\Gamma^J)$. If $H$ contains a matrix $g\notin\mathrm{CU}_{2n}(I:J^k,\Gamma_{\max}^{(I:J^k)})$, then $H$ contains $2n-3$ short root elements $\rho_{i1}(a)$ where $a\notin(I:J^{k-10})$ and $i$ takes values from $\{2,\ldots,2n\}$ except $n+1$, 3 or 2.*

**Proof.** When $\{(R_i, \Lambda_i)\}_{i\in I}$ is an inductive system of all finitely generated form subrings of $(R, \Lambda)$ with respect to the embeddings, one has $\mathrm{U}_{2n}(R, \Lambda) = \varinjlim \mathrm{U}_{2n}(R_i, \Lambda_i)$. Thus, one may assume that $R$ is Noetherian (replace $R$ by the ring generated by $s \in R_0\backslash M$, $d \in J$, and some matrix entries).

For every $M \in \mathrm{Max}(R_0)$, $(R_M, \Lambda_M)$ contains $(J_M, \Gamma^J_M)$ as a form ideal. The localization homomorphism $F_M : R \to R_M$ induces the following homomorphisms:

$$F_M : \mathrm{U}_{2n}(R, \Lambda) \to \mathrm{U}_{2n}(R_M, \Lambda_M),$$
$$F_M : \mathrm{CU}_{2n}(I : J^k, \Gamma^{(I:J^k)}_{\max}) \to \mathrm{CU}_{2n}(I_M : J^k_M, \Gamma^{(I_M:J^k_M)}_{\max}),$$
$$F_M : \mathrm{EU}_{2n}(J, \Gamma^J) \to \mathrm{EU}_{2n}(J_M, \Gamma^J_M).$$

By Lemma 4.2, we have that $F_M(g) \not\subseteq \mathrm{CU}_{2n}(I_M : J^k_M, \Gamma^{(I_M:J^k_M)}_{\max})$ for some $M \in \mathrm{Max}(R_0)$. Moreover, by Lemma 4.1, we can assume that ${}^{v_1}F_M(H)$, where $v_1 = \rho_{i1}(*)\rho_{j1}(*) \in \mathrm{E}^2(R_M)$ contains a matrix ${}^{v_1}F_M(g)$ whose first diagonal entry $g_{11}$ is invertible. Thus, ${}^{v_1}F_M(g)$ can be decomposed as ${}^{v_1}F_M(g) = uh$, where $u = \rho_{n+1,1}(*)\prod_{i=2}^{2n}\rho_{i1}(g_{i1}g_{11}^{-1}) \in \mathrm{EU}_{2n}(R_M, \Lambda_M)$ and $h$ is a matrix over $R_M$ of the following form:

$$\begin{pmatrix} h_{11} & * & \cdots & * & * & * & \cdots & * \\ 0 & & & & * & & & \\ \vdots & & A_1 & & \vdots & & B_1 & \\ 0 & & & & * & & & \\ 0 & 0 & \cdots & 0 & h_{11}^{*-1} & 0 & \cdots & 0 \\ 0 & & & & * & & & \\ \vdots & & C_1 & & \vdots & & D_1 & \\ 0 & & & & * & & & \end{pmatrix}. \tag{6.5}$$

Now, we may reduce the problem to the case $R_s$, where $s \in R_0\backslash M$; that is, ${}^{v_1}F_s(g) = uh$ where $v_1 = \rho_{i1}(*)\rho_{j1}(*) \in \mathrm{E}^2(R_s)$, $u = \prod_{i=2}^{2n}\rho_{i1}(*) \in \mathrm{EU}_{2n}(R_s, \Lambda_s)$, and $h$ is a matrix of form (6.5) over $R_s$. We subdivide the proof into two cases.

**Case 1.** $h$ lies in $\mathrm{CU}_{2n}(I_s : J^k_s, \Gamma^{(I_s:J^k_s)}_{\max})$. Because ${}^{v_1}F_s(g)$ does not lie in $\mathrm{CU}_{2n}(I_s : J^k_s, \Gamma^{(I_s:J^k_s)}_{\max})$, $u \notin \mathrm{CU}_{2n}(I_s : J^k_s, \Gamma^{(I_s:J^k_s)}_{\max})$. Without loss of generality, assume that the entry $u_{31}$ of $u$ in the position $(3, 1)$ is not in $(I_s : J^k_s)$. Taking $\rho_1 = \rho_{23}(s^p d_1)$, where $d_1 \in J$, satisfies that $s^p d_1 u_{31} \notin (I_s : J^{k-1}_s)$; thus, we have that ${}^{u^{-1}v_1}F_s(H)$ contains

$$g_1 = u^{-1}[\rho_1, {}^{v_1}F_s(g)]u = (u^{-1}\rho_1 u)(u^{-1}{}^{v_1}F_s(g)\rho_1^{-1}{}^{v_1}F_s(g)^{-1}u) = (u^{-1}\rho_1 u)(h\rho_1^{-1}h^{-1}),$$

which does not lie in $\mathrm{U}_{2n}(I_s : J^{k-1}_s, \Gamma^{(I_s:J^{k-1}_s)}_{\max})$ by Lemma 3.6 and that the first column of $g_1$ is of the form $(1, g'_{21}, 0, \ldots, 0, g'_{n+1,1}, 0, g'_{n+3,1}, 0, \ldots, 0)^T$. Note that the matrix $\rho_1 h\rho_1^{-1}h^{-1}$ lies in $\mathrm{U}_{2n}(I_s : J^{k-1}_s, \Gamma^{(I_s:J^{k-1}_s)}_{\max})$, and the entry $g'_{21}$ of $u^{-1}\rho_1 u\rho_1^{-1} = \rho_{21}(g'_{21})\rho_{n+3,1}(*)\rho_{n+1,1}(*)$ in the position $(2, 1)$ is not in $(I_s : J^{k-1}_s)$. Thus, the entry $g'_{21}$ of $g_1$ is not in $(I_s : J^{k-1}_s)$ either. Let $t_2 = \rho_{13}(1)\rho_{31}(-1)\rho_{13}(1)$. Hence, ${}^{t_2u^{-1}v_1}F_s(H)$ contains a matrix $g_1$ (denote ${}^{t_2}g_1$ still by $g_1$) whose entry $g'_{23}$ is not in $(I_s : J^{k-1}_s)$ and whose entries at the first row and the $(n + 1)$th column all lie in $(I_s : J^{k-1}_s)$ except $g'_{11}$ and $g'_{n+1,n+1}$, respectively. Denote $u^{-1}v_1$ as $t_1$, which is of the form $\prod_{i=2}^{2n}\rho_{i1}(*)$.

If $g'_{11}$ is not invertible, we go back to the case $R_M$. Consider the matrix $q_M = {}^{t_2}[\rho_{23}(s^p d_1), {}^{v_1}F_M(g)]$ for which the restriction on $R_s$ is $g_1$. Because the first row of $q_M$ is unimodular and $R_M$ is semi-local with at most two maximal ideals, we may choose a suitable matrix $v_2 = \rho_{l1}(*)\rho_{r1}(*) \in \mathrm{E}^2(R_M)$ such that the first diagonal entry of ${}^{v_2}q_M$ is invertible and the entry of ${}^{v_2}q_M$ in the position $(2, 3)$ is kept such that it does not lie in $(I_M : J^{k-1}_M)$ (note that the entries at the first row and the $(n + 1)$th column of $q_M$ all lie in $(I_M : J^{k-1}_M)$ except diagonal entries, so the entry of $q_M$ in the position $(2, 3)$ is still not in $(I_M : J^{k-1}_M)$ after adding some multiple of the non-diagonal entry at the first row or the $(n+1)$th column of $q_M$). Summarizing the above statements, we obtain that ${}^{t}F_M(H)$ $(t = v_2t_2t_1)$ contains a matrix $g_1$ (denote ${}^{v_2t_2}g_1$ still by $g_1$) whose first diagonal entry is invertible, the entry $g'_{23}$ in the position $(2, 3)$ does not lie in $(I_M : J^{k-1}_M)$, and the entries at the first row lie in $(I_M : J^{k-1}_M)$ except $g'_{11}$.

As shown above, factorize $g_1$ as $g_1 = u_1h_1$, where $u_1 = \prod_{i=2}^{2n}\rho_{i1}(*) \in \mathrm{EU}_{2n}(R_s, \Lambda_s)$, $h_1$ is a matrix of form (6.5) over $R_s$ for which the entry $g'_{23}$ in the position $(2, 3)$ is not in $(I_s : J^{k-1}_s)$ and the entries at the first row lie in $(I_s : J^{k-1}_s)$, except for $g'_{11}$. Taking $\rho_2 = \rho_{34}(s^p d_2)$, where $d_2 \in J$ satisfies that $s^p d_2 g'_{23} \notin (F_s(I) : F_s(J^{k-2})$, we obtain that ${}^{u_1^{-1}t}F_s(H)$ contains

$$g_2 = u_1^{-1}[\rho_2, g_1]u_1 = (u_1^{-1}\rho_2 u_1)(u_1^{-1}g_1\rho_2^{-1}g_1^{-1}u_1) \notin \mathrm{U}_{2n}(F_s(I) : F_s(J^{k-2}), \Gamma^{(F_s(I):F_s(J^{k-2}))}_{\max}),$$

whose first column is of the form $(1, 0, g''_{31}, 0, \ldots, 0, g''_{n+1,1}, 0, 0, g''_{n+4,1}, 0, \ldots, 0)^T$. We need to explain why there is at least one non-diagonal entry $g''_{2i}$ with $i \neq n + 1$ at the second row of $g_2$ not lying in $(F_s(I) : F_s(J^{k-2}))$ or why the diagonal entry $g''_{22}$ can be written as $1+\beta$ with $\beta \notin (F_s(I):F_s(J^{k-2}))$. Because the elements at the first row of $\rho_2 u_1^{-1} g_1 \rho_2^{-1} g_1^{-1} u_1 = \rho_2 h_1 \rho_2^{-1} h_1^{-1}$ all lie in $(F_s(I):F_s(J^{k-2}))$ except the first one, the entry of $\rho_2 h_1 \rho_2^{-1} h_1^{-1}$ in the position $(2, n+1)$ certainly lies in $(F_s(I):F_s(J^{k-2}))$. Then, by Remark 3.7(2), we obtain the conclusion for $\rho_2 h_1 \rho_2^{-1} h_1^{-1}$. It is clear that left multiplying $\rho_2 h_1 \rho_2^{-1} h_1^{-1}$ by $u_1^{-1}\rho_2 u_1 \rho_2^{-1}$ does not change the second row of $\rho_2 h_1 \rho_2^{-1} h_1^{-1}$. Therefore, the assertion holds for $g_2$.

Without loss of generality, assume that the entry $g''_{24}$ of $g_2$ in the position $(2, 4)$ does not lie in $(F_s(I):F_s(J^{k-2}))$. Taking $\rho_3 = \rho_{32}(s^p d_3)$, where $d_3 \in J$ satisfies that $s^p g''_{24} d_3 \notin (F_s(I):F_s(J^{k-3}))$, we have that ${}^{u_1^{-1}t}F_s(H)$ contains

$$g_3 = [\rho_3, g_2^{-1}] \notin \mathrm{U}_{2n}(F_s(I):F_s(J^{k-3}), \Gamma_{\max}^{(F_s(I):F_s(J^{k-3}))})$$

(note that the first column of $g_2$ is of the form $(1, 0, g''_{31}, 0, \ldots, 0, g'''_{n+1,1}, 0, 0, g'''_{n+4,1}, 0, \ldots, 0)^T$) and that the first column of $g_3$ is of the form $(1, 0, \ldots, 0, 0, \ldots, 0)^T$. Because all non-diagonal entries in the first row of $g_2^{-1}$ lie in $(F_s(I):F_s(J^{k-2}))$, the non-diagonal entries at the first row of $g_3$ lie in $(F_s(I):F_s(J^{k-3}))$. Thus, there exists a non-diagonal entry $g'''_{3i}$ with $i \neq n+1$ in the third row of $g_3$ not lying in $(F_s(I):F_s(J^{k-3}))$ or $g'''_{33}$ can be written as $1+\gamma$ with $\gamma \notin (F_s(I):F_s(J^{k-3}))$. Now, the matrix $g_3$ satisfies the condition of Corollary 5.5, and the matrix $t_3 = u_1^{-1} v_2$ is of the form $\prod_{i=2}^{2n} \rho_{i1}(*)$. Therefore, the matrices $t_1$, $t_2$, $t_3$ satisfy the condition of case (i) in Lemma 5.4. Applying Corollary 5.5, we conclude that $H$ contains $2n-3$ short root elements $\rho_{i1}(a)$, where $a \notin (I:J^{k-9})$, $i$ takes values from $\{2, \ldots, 2n\}$ except $n+1$ and 3.

**Case 2.** $h$ does not lie in $\mathrm{CU}_{2n}(I_s:J_s^k, \Gamma_{\max}^{(I_s:J_s^k)})$. Assume that there is a non-diagonal entry $h_{ij}$ of $h$ not lying in $(I_s:J_s^k)$. Let $\rho_1 = \rho_{jl}(s^p d_1)$, where $d_1 \in J$ satisfies that $s^p d_1 h_{ij} \notin (I_s:J_s^{k-1})$, and $l \neq i, \sigma i, \sigma j$. Thus, ${}^{u^{-1}v_1}F_s(H)$ contains a matrix

$$g_1 = u^{-1}[\rho_1, {}^{v_1}F_s(g)]u = (u^{-1}\rho_1 u)(u^{-1v_1}F_s(g)\rho_1^{-1v_1}F_s(g)^{-1}u) = (u^{-1}\rho_1 u)(h\rho_1^{-1}h^{-1})$$

where $g_1 \notin \mathrm{U}_{2n}(I_s:J_s^{k-1}, \Gamma_{\max}^{(I_s:J_s^{k-1})})$ and the first column of $g_1$ has the form $(1, 0, \ldots, 0, g'_{l1}, 0, \ldots, 0, g'_{n+1,1}, 0, \ldots, 0, g'_{\sigma j,1}, 0, \ldots, 0)^T$. By Remark 3.7(2), there is an entry $g'_{ir}$ with $r \neq 1,\ i$ in the $i$th row of $g_1$ satisfying $g'_{ir} \notin (I_s:J_s^{k-1})$. Otherwise, $g'_{ii} - 1 \notin (I_s:J_s^{k-1})$. Because $r \neq 1$, we may assume $r \neq i$ without a loss of generality. Taking $\rho_2 = \rho_{rm}(s^p d_2)$, where $d_2 \in J$ satisfies that $s^p d_2 g'_{ir} \notin (F_s(I):F_s(J^{k-2}))$, and $m \neq 1, i, j, \sigma j, n+1$, we conclude that ${}^{u^{-1}v_1}F_s(H)$ contains

$$g_2 = [\rho_2, g_1^{-1}] \notin \mathrm{U}_{2n}(F_s(I):F_s(J^{k-2}), \Gamma_{\max}^{(F_s(I):F_s(J^{k-2}))}),$$

whose first column is of the form $(1, 0, \ldots, 0, 0, \ldots, 0)^T$. Moreover, there is a non-diagonal entry $g''_{iq}$ in the $i$th row of $g_2$ satisfying that $g''_{iq} \notin (F_s(I):F_s(J^{k-2}))$; otherwise, $g''_{ii} - 1 \notin (F_s(I):F_s(J^{k-2}))$ by Remark 3.7(2).

When $i \neq 1$ and $q \neq n+1$, or even if $i = 1$ or $q = n+1$ there is another entry $g''_{xy}$ with $x \neq 1, y \neq n+1$ of $g_2$ not lying in $(F_s(I):F_s(J^{k-2}))$, in the case that matrix $g_2$ satisfies the condition of Corollary 5.5 and the matrix $t_1 = u^{-1}v_1$ is of the form $\prod_{i=2}^{2n} \rho_{i1}(*)$, we may apply Corollary 5.5 under the condition that $t_2 = t_3 = 1_{2n}$ to obtain that $H$ contains $2n-2$ short root elements $\rho_{i1}(a)$ where $a \notin (I:J^{k-8})$, $i$ takes values from $\{2, \ldots, 2n\}$ except $n+1$.

When $i = 1$ or $q = n+1$ and all other entries not lying in the first row and the $(n+1)$th column of $g_2$ belong to $(F_s(I):F_s(J^{k-2}))$, we assume that $g''_{13} \notin (F_s(I):F_s(J^{k-2}))$ without loss of generality. Let $t_2 = \rho_{12}(-1)\rho_{21}(1)$. Thus, ${}^{t_2 u^{-1} v_1}F_s(H)$ contains a matrix $g_2$ (denote ${}^{t_2}g_2$ still by $g_2$) whose entry $g''_{23}$ is not in $(F_s(I):F_s(J^{k-2}))$ and whose entries in the first row lie in $(F_s(I):F_s(J^{k-2}))$ except for $g''_{11}$ and $g''_{1,n+2}$ (this may occur if the entry of $g_2$ in the position $(1, n+2)$ is not in $(F_s(I):F_s(J^{k-2}))$). If $g''_{11}$ is not invertible, we go back to the case $R_M$. Copying the proof of Case 1 to the same situation, we have that ${}^{v_2 t_2 t_1}F_M(H)$ ($v_2 = \rho_{h1}(*)\rho_{k1}(*)$, $t_1 = u^{-1}v_1$) contains a matrix $g_2$ such that the first diagonal entry is invertible, the entry $g''_{23}$ in the position $(2, 3)$ does not lie in $(F_M(I):F_M(J^{k-2}))$, and the entries at the first row lie in $(F_M(I):F_M(J^{k-2}))$, except for $g''_{11}$ and $g''_{1,n+2}$. The rest of the proof is quite similar to Case 1; therefore, we will only provide a sketch.

As in Case 1, write $g_2 = u_2 h_2$, where $u_2 = \prod_{i=2}^{2n} \rho_{i1}(*) \in \mathrm{EU}_{2n}(R_s, \Lambda_s)$, $h_2$ is a matrix of form (6.5) over $R_s$ for which the entry $g''_{23}$ in the position $(2, 3)$ is not in $(I_s:J_s^{k-2})$, and the entries at the first row lie in $(I_s:J_s^{k-2})$, except for $g''_{11}$ and $g''_{1,n+2}$. Taking $\rho_3 = \rho_{34}(s^p d_3)$, where $d_3 \in J$ satisfies that $s^p d_3 g''_{23} \notin (F_s(I):F_s(J^{k-3}))$, we obtain that ${}^{u_2^{-1}t}F_s(H)$ ($t = v_2 t_2 t_1$) contains

$$g_3 = u_2^{-1}[\rho_3, g_2]u_2 \notin \mathrm{U}_{2n}(F_s(I):F_s(J^{k-3}), \Gamma_{\max}^{(F_s(I):F_s(J^{k-3}))}).$$

The matrix $g_3$ has the same form and properties as the matrix $g_2$ in Case 1, except for the level of principal congruence subgroup to which they do not belong, *i.e.*, $g_3 \notin \mathrm{U}_{2n}(F_s(I):F_s(J^{k-3}), \Gamma_{\max}^{(F_s(I):F_s(J^{k-3}))})$, whereas $g_2 \notin \mathrm{U}_{2n}(F_s(I):F_s(J^{k-2}), \Gamma_{\max}^{(F_s(I):F_s(J^{k-2}))})$ in Case 1. Following the proof of Case 1, we obtain that ${}^{t_3 t_2 t_1}F_s(H)$, where $t_3 = u_2^{-1}v_2$, contains a matrix $g_4$ that satisfies the condition of Corollary 5.5. Meanwhile, the matrices $t_1, t_2, t_3$ satisfy the condition of case (ii) in Lemma 5.4. Applying Corollary 5.5, we have that $H$ contains $2n-3$ short root elements $\rho_{i1}(a)$, where $a \notin (I:J^{k-10})$, $i$ takes values from $\{2, \ldots, 2n\}$, except for $n+1$ and 2. □

## 7. Proof of main theorem

Applying Proposition 6.3 and Lemma 6.1, we can obtain the following theorem, called the weak structure theorem.

**Theorem 7.1.** *Let $H$ be a subgroup of $\mathrm{U}_{2n}(R, \Lambda)$, $n \geq 4$, normalized by $\mathrm{EU}_{2n}(J, \Gamma^J)$. Suppose that $(I, \Gamma^I)$ is the largest form ideal of $(R, \Lambda)$ with the property that $\mathrm{EU}_{2n}(I, \Gamma^I) \subseteq H$. Thus, $H \subseteq \mathrm{CU}_{2n}(I : J^{12}, \Gamma_{\max}^{(I:J^{12})})$.*

**Proof.** Suppose that $H \nsubseteq \mathrm{CU}_{2n}(I : J^{12}, \Gamma_{\max}^{(I:J^{12})})$. By Proposition 6.3, $H$ contains $2n-3$ short root elements $\rho_{i1}(a)$, where $a \notin (I : J^2)$ and $i$ takes values from $\{2, \ldots, 2n\}$ except $n+1$, 3 or 2. Thus, by Lemma 6.1, $\mathrm{EU}_{2n}(aJ^2 + a^*J^2, \Gamma_{\min}^{(aJ^2+a^*J^2)}) \subseteq H$. However, $a \notin (I : J^2)$, we have that $aJ^2 + a^*J^2 \nsubseteq I$. Therefore, $\mathrm{EU}_{2n}(I, \Gamma^I) \subset \mathrm{EU}_{2n}(I + aJ^2 + a^*J^2, \Gamma^I + \Gamma_{\min}^{(aJ^2+a^*J^2)}) \subseteq H$. This is a contradiction. □

To prove the main theorem, we need the following lemma.

**Lemma 7.2.** *Under the assumptions of Theorem 7.1, let $s \in R_0 \backslash M$, $M \in Max(R_0)$, and let $g = \rho_{n+1,1}(\xi) \prod_{i \neq n+1} \rho_{i1}(a_i)$, where $\xi \in \Gamma_{\max}^{I_s}$, $a_i \in I_s$. Suppose that $F_s(H)$ contains $[g, \rho_{1j}(s^N d)]$, where $d \in J$ and $N$ is a sufficiently large integer ($N \geq 2p$). Thus, $F_s(H)$ contains $\rho_{\sigma j,j}(\lambda^{-(\epsilon(\sigma j)+1)/2} s^N d^* \xi d s^N)$ where $j \neq 1, n+1$.*

**Proof.** Applying the commutator identity $[uv, w] = [u, w]^{wu}[w^{-1}, v]$, we have

$$[g, \rho_{1j}(s^N d)] = [\rho_{n+1,1}(\xi), \rho_{1j}(s^N d)] \cdot {}^{\rho_{1j}(s^N d)\rho_{n+1,1}(\xi)} \left[ \rho_{1j}(-s^N d), \prod_{i \neq n+1} \rho_{i1}(a_i) \right]. \tag{7.1}$$

It is clear that $[\rho_{n+1,1}(\xi), \rho_{1j}(s^N d)] = \rho_{n+1,j}(s^N d\xi) \rho_{\sigma j,j}(-\lambda^{(\epsilon(j)-1)/2} s^N d^* \xi d s^N)$ (note that $s^* = s$) and the first factor $\rho_{n+1,j}(s^N d\xi)$ lies in $\mathrm{EU}_{2n}(F_s(I), F_s(\Gamma^I))$. Let us consider the second factor in (7.1). Denote $\rho_{1j}(s^N d)\rho_{n+1,1}(\xi)$ by $x$. As above,

$$\left[ \rho_{1j}(-s^N d), \prod_{i \neq n+1} \rho_{i1}(a_i) \right] = \left[ \prod_{i \neq n+1} \rho_{i1}(a_i), \rho_{1j}(-s^N d) \right]^{-1}$$
$$= {}^y \left[ \prod_{i \neq j, n+1} \rho_{i1}(a_i), \rho_{1j}(-s^N d) \right] \cdot [\rho_{1j}(-s^N d), \rho_{j1}(a_j)]$$

where $y = \rho_{1j}(-s^N d)\rho_{j1}(a_j)$. Direct computation shows that $\eta = \left[ \prod_{i \neq j, n+1} \rho_{i1}(a_i), \rho_{1j}(-s^N d) \right] \in \mathrm{EU}_{2n}(s^N I_s, s^N \Gamma^{I_s})$. Because for sufficiently large $N$, one has $g\mathrm{EU}_{2n}(s^N I_s, s^N \Gamma^{I_s})g^{-1} \subseteq \mathrm{EU}_{2n}(F_s(I), F_s(\Gamma^I))$ for a given $g \in \mathrm{EU}_{2n}(R_s, \Lambda_s)$ (see [11]), ${}^{xy}\eta \in \mathrm{EU}_{2n}(F_s(I), F_s(\Gamma^I))$. By assuming that $F_s(H)$ contains $\mathrm{EU}_{2n}(F_s(I), F_s(\Gamma^I))$, if we show that ${}^x[\rho_{1j}(-s^N d), \rho_{j1}(a_j)] \in \mathrm{EU}_{2n}(F_s(I), F_s(\Gamma^I))$ then the proof will be finished. Using the commutator identity $\rho_{1j}(-s^N d) = [\rho_{1i}(-s^{\frac{N}{2}} d), \rho_{ij}(s^{\frac{N}{2}})]$ and using the Hall-Witt identity ${}^u[[v, u^{-1}], w] = {}^v[u, [v^{-1}, w]] \cdot {}^w[v, [w^{-1}, u]]$, it suffices to show that $[\rho_{1i}(-s^{\frac{N}{2}} d), \rho_{j1}(a_j)]$ and $[\rho_{ij}(s^{\frac{N}{2}}), \rho_{j1}(a_j)]$ lie in $\mathrm{EU}_{2n}(s^{N/2} I_s, s^{N/2} \Gamma^{I_s})$. However, these are obvious. □

Before proving the main theorem, we point out one important fact: in Lemma 3.5, if $a \in J$, $g_{lh} \in (I : J)$ then $a^* g_{lh} |v_{g(i)}|_q g_{lh}^* a \in \Gamma_{\min}^I \subseteq \Gamma^I$, where $(I, \Gamma^I)$ is a form ideal of $(R, \Lambda)$. When a matrix $g \in \mathrm{U}_{2n}(I : J, \Gamma_{\max}^{(I:J)})$, all non-diagonal entries of $g$ lie in $(I : J)$. Thus, for diagonal entry $g_{ll}$ we have that $a^* g_{ll} |v_{g(i)}|_q g_{ll}^* a = a^* |v_{g(i)}|_q a \pmod{\Gamma^I}$ because the rows and columns of $g$ are unimodular.

Now, let us complete the proof of Theorem 1. For convenience, denote $\Omega(J, I, \Gamma^I, 12, 14)$ by $\Omega$.

By Theorem 7.1, $H \subseteq \mathrm{CU}_{2n}(I : J^{12}, \Gamma_{\max}^{(I:J^{12})})$. Suppose that $H \nsubseteq \mathrm{CU}_{2n}(I : J^{12}, \Omega)$. Thus, there exists a matrix $g \in H$ which has at least one column, say $v_2$, whose length $|v_{g(2)}|_q$ is not in $\Omega$. Because $g \in \mathrm{CU}_{2n}(I : J^{12}, \Gamma_{\max}^{(I:J^{12})})$, we have that $g_1 = [g, \rho_{21}(d_1)] \in H \bigcap \mathrm{U}_{2n}(I, \Gamma_{\max}^I)$ for all $d_1 \in J^{12}$. Note that there is only the diagonal entry $g'_{11}$ in the first row of $g^{-1}$, which does not lie in $(I : J^{12})$, and the others, non-diagonal entries, all lie in $(I : J^{12})$. Investigating the computation of the commutator $[g, \rho_{21}(d_1)]$, we conclude that the length of the first column of $g_1$ does not lie in $\Gamma^I$ and the lengths of other columns of $g_1$ lie in $\Gamma^I$, except for the $(n+2)$th column, which may not be in $\Gamma^I$ if the length of the $(n+1)$th column of $g$ does not lie in $\Omega$.

By Lemma 4.2, $F_M(g_1) \nsubseteq \mathrm{U}_{2n}(I_M, \Gamma_M^I)$ for some $M \in \mathrm{Max}(R_0)$. Choose a suitable $t = \rho_{1k}(a)\rho_{1r}(b) \in \mathrm{E}^2(R_M)$ such that the first diagonal entry of ${}^tF_M(g_1)$ is invertible and the length of the first column of $F_M(g_1)$ is maintained. Thus, ${}^tF_M(g_1)$ can be decomposed as ${}^tF_M(g_1) = uh$, where $u = \rho_{n+1,1}(\sum g_{11}^{-1*} g_{i1}^* g_{n+i,1} g_{11}^{-1} + g_{n+1,1} g_{11}^{-1}) \cdot \prod_{i \neq n+1} \rho(g_{i1} g_{11}^{-1})$ (here all $g_{i1}$ are elements in $R_M$) and $h$ is a matrix of form (6.5) over $R_M$. Note that $g_{11}^*(\sum g_{11}^{-1*} g_{i1}^* g_{n+i,1} g_{11}^{-1} + g_{n+1,1} g_{11}^{-1}) g_{11} = (|v_{g_1(1)}|_q)_M$. Because $g_{11}$ is invertible, we have $\sum g_{11}^{-1*} g_{i1}^* g_{n+i,1} g_{11}^{-1} + g_{n+1,1} g_{11}^{-1} = (|v_{g_1(1)}|_q)_M$ and denote it by $\xi$.

As in the proof of Proposition 6.3, reduce the problem to the case $R_s$ where $s \in R_0 \backslash M$; that is, ${}^tF_s(g_1) = uh$ where $u = \rho_{n+1,1}(\xi) \prod_{i \neq n+1} \rho_{i1}(*) \in \mathrm{EU}_{2n}(R_s, \Lambda_s)$, $\xi = (|v_{g_1(1)}|_q)_s$ and $h$ is a matrix of form (6.5) over $R_s$. Taking $\rho_1 = \rho_{1j}(s^N d_1)$, where $d_1 \in J, j \neq n+1, 2$ (note that the length of the $(n+2)$th column of ${}^tF_s(g_1)$ may be not in $\Gamma_s^I$), and $N$ is a sufficiently large integer, we have that ${}^tF_s(H)$ contains

$$g_2 = \left[\rho_{n+1,1}(\xi)\prod_{i\neq n+1}\rho_{i1}(*)h,\ \rho_1\right] = \left[\rho_{n+1,1}(\xi)\prod_{i\neq n+1}\rho_{i1}(*),\ \rho_1\right]\cdot {}^x\left[\rho_1^{-1}, h\right]$$

where $x = \rho_1 \cdot \rho_{n+1,1}(\xi)\prod_{i\neq n+1}\rho_{i1}(*)$. Because the lengths of the rows of $h^{-1}$ all lie in $\Gamma_{\min}^{I_s}$, except for the second, it is easy to show that the second factor lies in $\mathrm{EU}_{2n}(F_s(I), \Gamma_{\min}^{F_s(I)}) \subseteq {}^tF_s(H)$. Thus, by Lemma 7.2, we have that ${}^tF_s(H)$ contains

$$\rho_{\sigma j,j}(\lambda^{-(\epsilon(\sigma j)+1)/2}s^N d_1^*\xi d_1 s^N)$$

where $j \neq 1, 2, n+1$.

There are $2n-7$ choices for the lower index $j$ of $\rho_{\sigma j,j}(\lambda^{-(\epsilon(\sigma j)+1)/2}s^N d_1^*\xi d_1 s^N)$ such that it commutes with $t = \rho_{1k}(a)\rho_{1r}(b)$ ($j$ should be not equal to $1, 2, k, \sigma k, r, \sigma r, n+1$). When $n = 4$, we only have one choice for $j$. Thus, suppose that $t$ does not commute with $\rho_{k,\sigma k}(\lambda^{-(\epsilon(k)+1)/2}s^N d_1^*\xi d_1 s^N)$. Then $F_s(H)$ contains

$$\begin{aligned}&{}^{t^{-1}}\rho_{k,\sigma k}(\lambda^{-(\epsilon(k)+1)/2}s^N d_1^*\xi d_1 s^N)\\ =&\rho_{1,\sigma k}(-\lambda^{-(\epsilon(k)+1)/2}as^N d_1^*\xi d_1 s^N)\rho_{1,n+1}(*)\rho_{k,\sigma k}(\lambda^{-(\epsilon(k)+1)/2}s^N d_1^*\xi d_1 s^N).\end{aligned}$$

Because the first factor above lies in $F_s(\mathrm{EU}_{2n}(I, \Gamma^I)) \subseteq F_s(H)$, $F_s(H)$ contains $\rho_{1,n+1}(*)\cdot\rho_{k,\sigma k}(\lambda^{-(\epsilon(k)+1)/2}s^N d_1^*\xi d_1 s^N)$. Let $\eta \in H$ such that $F_s(\eta) = \rho_{1,n+1}(*)\rho_{k,\sigma k}(\lambda^{-(\epsilon(k)+1)/2}s^N d_1^*\xi d_1 s^N)$. By the argument made in Lemma 5.3 we have that $H$ contains

$$\rho_{i,\sigma k}(\lambda^{-(\epsilon(k)+1)/2}s^p d_2 s^N d_1^*\xi d_1 s^N)\rho_{i,\sigma i}(\lambda^{-(\epsilon(i)+1)/2}s^{p+N}d_2 d_1^*\xi d_1 d_2^* s^{p+N}) = \left[\rho_{ik}(s^p d_2), \eta\right]$$

and hence contains $\rho_{i,\sigma i}(\lambda^{-(\epsilon(i)+1)/2}s^{p+N}d_2 d_1^*\xi d_1 d_2^* s^{p+N})$, where $k \neq 1, 2, n+1$, but $i$ can go over from 1 to $2n$ ($k$ may take values from 1 to $2n$ except the three values above). Set

$$C = \{s \in R_0 | \rho_{i,\sigma i}(\lambda^{-(\epsilon(i)+1)/2}sd_2 d_1^*\xi d_1 d_2^* s) \in H, \xi \in \Gamma_{\max}^{(I:J^{12})}, d_1, d_2 \in J\}.$$

It is clear that $C$ is an ideal of $R_0$ and that $C$ cannot be contained in any maximal ideal of $R_0$. Hence, $C = R_0$. This implies that all long root elements $\rho_{i,\sigma i}(\lambda^{-(\epsilon(i)+1)/2}d_2 d_1\xi d_1^* d_2^*) \in H$ where $\xi \in \{d\gamma d^* | \gamma \in \Gamma_{\max}^{(I:J^{12})}, d \in J^{12}\}$. It is not difficult to show that $\mathrm{EU}_{2n}(I, \Gamma^I + \{d_2 d_1\xi d_1^* d_2^*\}) \subseteq H$ (see Lemma 2.7 in [17]). Thus, if $d_2 d_1 \xi d_1^* d_2^* \notin \Gamma^I$, it will lead to the contradiction that $(I, \Gamma^I)$ is the largest form ideal of $(R, \Lambda)$ with the property that $\mathrm{EU}_{2n}(I, \Gamma^I) \subseteq H$. We complete the proof.

**Corollary 7.3.** *Let $H$ be a subgroup of $\mathrm{U}_{2n}(R, \Lambda)$, $n \geq 4$, normalized by $\mathrm{EU}_{2n}(J, \Gamma^J)$. Suppose that $(I, \Gamma^I)$ is the smallest form ideal of $(R, \Lambda)$ with the property that $H \subseteq \mathrm{CU}_{2n}(I, \Gamma^I)$. Thus,*

$$\mathrm{EU}_{2n}(IJ^{12}, \Delta(J, I, \Gamma^I, 12, 14)) \subseteq H,$$

*where $\Delta(J, I, \Gamma^I, k, l) = \Gamma_{\min}^{(IJ^k)} + \{r^*\zeta r | \zeta \in \Gamma^I, r \in J^l, k \leq l\}$.*

**Proof.** The proof is same as that of Corollary 2.9 in [17]. For the sake of completeness, let us sketch the proof.

Let $(B, \Gamma^B)$ be the largest form ideal of $(R, \Lambda)$ such that $\mathrm{EU}_{2n}(B, \Gamma^B) \subseteq H$. By Theorem 1,

$$H \subseteq \mathrm{CU}_{2n}(B : J^{12}, \Omega(J, B, \Gamma^B, 12, 14)).$$

Because $(I, \Gamma^I)$ is the smallest form ideal such that $H \subseteq \mathrm{CU}_{2n}(I, \Gamma^I)$, we have

$$I \subseteq (B : J^{12}) \text{ and } \Gamma^I \subseteq \Omega(J, B, \Gamma^B, 12, 14).$$

The first inclusion implies that $IJ^{12} \subseteq B$ and $\Gamma_{\min}^{(IJ^{12})} \subseteq \Gamma^B$. Let $x \in \{r^*\zeta r | \zeta \in \Gamma^I, r \in J^{14}\}$. The second inclusion above implies that $x \in \Gamma^B$. Thus,

$$\Delta(J, I, \Gamma^I, 12, 14) \subseteq \Gamma^B.$$

The proof is completed. □

## 8. The structure of subnormal subgroups

In this section we discuss the structure of subnormal subgroups of $\mathrm{U}_{2n}(R, \Lambda)$. The definition of subnormal subgroup of a group $G$ has been given in Section 1 (see (1.1)).

Let $H$ be a subgroup of $G \subseteq \mathrm{U}_{2n}(R, \Lambda)$. We use $L(H)$ to denote the maximal form ideal $(I, \Gamma^I)$ of form ring $(R, \Lambda)$ such that $\mathrm{EU}_{2n}(I, \Gamma^I) \subseteq H$. Similarly $U(H)$ denotes the smallest form ideal $(B, \Gamma^B)$ of form ring $(R, \Lambda)$ such that $H \subseteq \mathrm{CU}_{2n}(B, \Gamma^B)$. Furthermore, we define the following operations on form ideals. For two form ideals $(I, \Gamma^I)$, and $(J, \Gamma^J)$, and for two non-negative integer $k$ and $l$ with $k \le l$, define

$$\big((I, \Gamma^I), (J, \Gamma^J), (k, l)\big) = (IJ^k, \triangle(J, I, \Gamma^I, k, l)),$$

where $\triangle(J, I, \Gamma^I, k, l)$ is defined in Corollary 7.3. Setting $J^0 = R$, one obtains

$$\big((I, \Gamma^I), (J, \Gamma^J), (0, 0)\big) = \big(I, \{r^*\xi r | \xi \in \Gamma^I, r \in R\}\big) \subseteq (I, \Gamma^I).$$

For any $k, l \ge 1$, denote

$$\big((J, \Gamma^J), (J, \Gamma^J), (k-1, l-1)\big) = (J, \Gamma^J)^{(k,l)}.$$

Set $(J, \Gamma^J)^{(0,0)} = (R, \Lambda)$ for any form ideal $(J, \Gamma^J)$. Thus, we have the following property

$$\big((I, \Gamma^I), (J, \Gamma^J), (k_1k_2, l_1l_2)\big) \subseteq ((I, \Gamma^I), (J, \Gamma^J)^{(k_1,l_1)}, (k_2, l_2)).$$

**Remark 8.1.** The operation on two form ideals $(I, \Gamma^I)$ and $(J, \Gamma^J)$ when $k = l = 1$ specializes the definition of their product. Reviewing the proofs of main theorem and related lemmas, one may find that we only use short root elements of level $(J, \Gamma^J)$ to performing commutator calculus because if $\Lambda = 0$ there is no long root element in $\mathrm{EU}_{2n}(R, \Lambda)$. That is, we only use the assumption that $H$ is normalized by $\mathrm{EU}_{2n}(J, \Gamma^J_{\min})$ to prove the main result even though $H$ is supposed to be normalized by $\mathrm{EU}_{2n}(J, \Gamma^J)$. In that case, the last term $\{b^*ab \mid b \in I, a \in \Gamma^J\}$ in Definition 2.1 lies in $\Gamma^{IJ}_{\min}$.

The following theorem is an analog of Theorem 2.11 in [17] for a stable unitary group; the proof easily follows from Theorem 1 and Corollary 7.3 by an argument similar to that made in the proof of Theorem 2.11 in [17].

**Theorem 8.2.** *Let $G$ be a subgroup of $\mathrm{U}_{2n}(R, \Lambda)$ ($n \ge 4$) containing $\mathrm{EU}_{2n}(R, \Lambda)$, and let $H$ be a subnormal subgroup of $G$, i.e., $H \lhd^d G$ for some integer $d$. Thus, $U(H)^{(k(d),l(d))} \subseteq L(H)$, where $k(d) = (12^d - 1)/11$, $l(d) = (14^d - 1)/13$.*


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